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Equivalence classes of related evolution equations and Lie symmetries

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Abstract. This is an extension of earlier work by the authors which gives a correspondence between Lie symmetry operators (with non-trivial time dependence) for a given evolution equation and those evolution equations related to the given one by a change of independent and dependent coordinates. Here we work out the correspondence between symmetries of a system of evolution equations $\boldsymbol{v}_t = \boldsymbol{K}(y, \boldsymbol{v})$ and those systems $\boldsymbol{u}_s = \boldsymbol{J}(x, \boldsymbol{u})$ related to it by a change of coordinates $t = T(s, x, \boldsymbol{u})$, $y = Y(s, x, \boldsymbol{u})$, $\boldsymbol{v} = \boldsymbol{V}(s, x, \boldsymbol{u})$ and show (extending ideas of Humi and Rosencrans) how to determine the related equations directly from the symmetry operators without solving a system of differential equations. In general there are multiple evolution equations associated with a given symmetry; for the case of scalar evolution equations we compute explicitly the structure of each equivalence class.

1. Introduction

An *evolution equation* is a system of $m \geq 1$ partial differential equations of the form

$$(*) \qquad \boldsymbol{\Omega} \equiv \boldsymbol{v}_t - \boldsymbol{K}(y, \boldsymbol{v}, \boldsymbol{v}_{i_1 \ldots i_n}) = \boldsymbol{0} \tag{1.1}$$

where

$$\boldsymbol{v} = (v^1, \ldots, v^m) \qquad y = (y^1, \ldots, y^n)$$
$$v^l_t = \partial_t v^l \qquad v^l_{i_1 \ldots i_n} = \partial^{i_1}_{y^1} \ldots \partial^{i_n}_{y^n} v^l$$

and $\boldsymbol{K} = (K^1, \ldots, K^m)$ depends on only a finite number $M > 0$ of the derivatives $v^l_{i_1 \ldots i_n}$. We assume that $\boldsymbol{K}$ is a local analytic function of its $n + m + M$ variables and, on occasion, that it is a polynomial or rational function of the derivatives $v^l_{i_1 \ldots i_n}$, $i_1 + \ldots + i_n > 0$. Furthermore, we assume that the system (1.1) is locally solvable in the sense of Olver [1, p 162]. A solution of (1.1) is a function $\boldsymbol{v} = \boldsymbol{v}(t, y)$, analytic in the variables (t, y) such that (1.1) is well defined and identically satisfied for all $(t, y) \in S$ where S is a non-empty open set in C^{n+1}. (In the following all functions are assumed to be locally analytic.) A second evolution equation

$$(+) \qquad \boldsymbol{\Phi} \equiv \boldsymbol{u}_s - \boldsymbol{J}(x, \boldsymbol{u}, \boldsymbol{u}_{j_1 \ldots j_n}) = \boldsymbol{0}$$
$$\boldsymbol{u} = (u^1, \ldots, u^m) \qquad x = (x^1, \ldots, x^n) \tag{1.2}$$

is *related* to $(*)$ if there is a coordinate transformation

$$t = T(s, x, \boldsymbol{u}) \qquad y = \boldsymbol{Y}(s, x, \boldsymbol{u}) \qquad \boldsymbol{v} = \boldsymbol{V}(s, x, \boldsymbol{u}) \tag{1.3}$$

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which maps (*) to (+). Here we assume that the Jacobian $\det(\partial(t, \boldsymbol{y}, \boldsymbol{v})/\partial(s, \boldsymbol{x}, \boldsymbol{u}))$ is locally non-zero and limit ourselves to solutions $\boldsymbol{u}(s, \boldsymbol{x})$ of (+) such that

$$\det(\partial(T(s, \boldsymbol{x}, \boldsymbol{u}(s, \boldsymbol{x})), \boldsymbol{Y}(s, \boldsymbol{x}, \boldsymbol{u}(s, \boldsymbol{x})))/\partial(s, \boldsymbol{x}))$$

is locally non-zero. Thus, given $\boldsymbol{u}(s, \boldsymbol{x})$ we can solve the equations $t = T$, $\boldsymbol{y} = \boldsymbol{Y}$ for $s, \boldsymbol{x}$ and then substitute this result into $\boldsymbol{v} = \boldsymbol{V}$ to obtain $\boldsymbol{v} = \boldsymbol{f}(t, \boldsymbol{y})$. Similarly equations (1.3) can be inverted to express $s, \boldsymbol{x}, \boldsymbol{u}$ as functions of $t, \boldsymbol{y}, \boldsymbol{v}$ and, in particular, to transform a solution $\boldsymbol{v}(t, \boldsymbol{y})$ of (*) into a solution $\boldsymbol{u} = \boldsymbol{h}(s, \boldsymbol{x})$ of (+).

It is evident that an arbitrary coordinate transformation of the form

$$t = s \qquad \boldsymbol{y} = \boldsymbol{Y}(\boldsymbol{x}) \qquad \boldsymbol{v} = \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{u}) \tag{1.4}$$

will map (*) to a related evolution equation, so we consider such transformations as *trivial*. Furthermore, the more general transformation

$$t = s \qquad \boldsymbol{y} = \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{u}) \qquad \boldsymbol{v} = \boldsymbol{V}(\boldsymbol{x}, \boldsymbol{u}) \tag{1.5}$$

with $\det(\partial \boldsymbol{Y}(\boldsymbol{x}, \boldsymbol{u}(s, \boldsymbol{x}))/\partial \boldsymbol{x}) \neq 0$ may map (*) to a related evolution equation if it preserves the polynomial nature of the evolution equation. In addition, transformations of the form

$$t = s + \varphi(\boldsymbol{x}, \boldsymbol{u}) \qquad \boldsymbol{y} = \boldsymbol{x} \qquad \boldsymbol{v} = \boldsymbol{u} \tag{1.6}$$

(with the property that $\partial_t = \partial_s$) may sometimes map an evolution equation to another evolution equation but we shall show that such mappings with $\varphi_{\boldsymbol{x}} \neq 0$ or $\varphi_{\boldsymbol{u}} \neq 0$ can occur only for very restricted classes of evolution equations. Our interest is in determining all equivalence classes of evolution equations related to a given equation through transformations (1.3) with $T_s = \partial_s T(s, \boldsymbol{x}, \boldsymbol{u}) \neq 0$, where two evolution equations are equivalent if they are related by the composition of a coordinate transformation (1.5) and a transformation (1.6). Also we will investigate the conditions under which an evolution equation admits the non-trivial transformation (1.5) or (1.6). In [2] this problem was essentially solved for the special case of scalar evolution equations and transformations (1.3) such that $T_u = Y_u = 0$. Here we extend an idea found in [3] and [4] to treat the general case.

As is well known [1], every generalised Lie symmetry of (1.1) can be expressed in the standard form

$$X(\boldsymbol{f}) = \boldsymbol{f} \cdot \partial_{\boldsymbol{v}} + D_t \boldsymbol{f} \cdot \partial_{\boldsymbol{v}_t} + \sum_{i_1 + \ldots + i_n \geq 1} D_{y^1}^{i_1} \ldots D_{y^n}^{i_n} \boldsymbol{f} \cdot \partial_{\boldsymbol{v}_{i_1 \ldots i_n}} \tag{1.7}$$

where $\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{v}, \boldsymbol{v}_{i_1 \ldots i_n})$ is an m-tuple and D_t, D_{y^k} are total derivatives, e.g.,

$$D_{y^k} = \partial_{y^k} + \boldsymbol{v}_{y^k} \cdot \partial_{\boldsymbol{v}} + \boldsymbol{v}_{ty^k} \cdot \partial_{\boldsymbol{v}_t} + \sum_{i_1 + \ldots + i_n \geq 1} \boldsymbol{v}_{i_1 \ldots i_k + 1 \ldots i_n} \cdot \partial_{\boldsymbol{v}_{i_1 \ldots i_k \ldots i_n}}. \tag{1.8}$$

In particular, $X(\boldsymbol{f})$ is a generalised Lie symmetry provided

$$X(\boldsymbol{f})\boldsymbol{\Omega} = 0$$

whenever $\boldsymbol{\Omega} \equiv \boldsymbol{0}$. Special Lie symmetries are the point symmetries of the form

$$Y = \tau(t, \boldsymbol{y}, \boldsymbol{v})\partial_t + \boldsymbol{\xi}(t, \boldsymbol{y}, \boldsymbol{v}) \cdot \partial_{\boldsymbol{y}} + \boldsymbol{\eta}(t, \boldsymbol{y}, \boldsymbol{v}) \cdot \partial_{\boldsymbol{v}}. \tag{1.9}$$

These operators correspond to standard form operators $X(\boldsymbol{f})$ [1, 5] where

$$\boldsymbol{f} = \boldsymbol{\eta} - \sum_{j=1}^{n} \xi^j \boldsymbol{v}_{y^j} - \tau \boldsymbol{K}. \tag{1.10}$$

In § 2 we demonstrate that there is a one-to-one association between equivalence classes of evolution equations related to (*) and point symmetry operators (1.9) for (*) such that $\tau \neq 0$. The demonstration will include the explicit construction of an evolution equation in each equivalence class. In § 3 we determine conditions under which an evolution equation which is either a polynomial, rational or general analytic function of its spatial derivatives, will admit a non-trivial coordinate transformation (1.5) or (1.6). We show that for equations of physical interest these transformations are seldom admitted, so that each equivalence class consists only of equations related by trivial coordinate transformations (1.4). In particular we will show that no *scalar* polynomial evolution equation admits a non-trivial transformation (1.5) if the equation (1.1) depends non-trivially on at least one spatial derivative $v_{i_1 \ldots i_n}$ with $i_1 + \ldots + i_n \geq 2$; furthermore the equation admits no transformation (1.6) with $\varphi_u \neq 0$. Moreover, unless the equation is degenerate in a certain precise sense it will admit no transformation (1.6) with $\varphi_x \neq 0$.

2. The fundamental relationship

Theorem 1. Let

$$\boldsymbol{\Omega} \equiv \boldsymbol{v}_t - \boldsymbol{K}(\boldsymbol{y}, \boldsymbol{v}, \boldsymbol{v}_{i_1 \ldots i_n}) = \boldsymbol{0} \tag{2.1}$$

be an evolution equation. There is a one-to-one correspondence between (equivalence classes of) evolution equations related to $\boldsymbol{\Omega} = \boldsymbol{0}$ via coordinate transformations (1.3) with $T_s \neq 0$ and point symmetry operators for $\boldsymbol{\Omega} = \boldsymbol{0}$ of the form

$$Y = \tau(t, \boldsymbol{y}, \boldsymbol{v})\partial_t + \boldsymbol{\xi}(t, \boldsymbol{y}, \boldsymbol{v}) \cdot \partial_{\boldsymbol{y}} + \boldsymbol{\eta}(t, \boldsymbol{y}, \boldsymbol{v}) \cdot \partial_{\boldsymbol{v}} \tag{2.2}$$

with $\tau \neq 0$.

Proof. First of all we note that $\partial_t = \partial_s$ if and only if the coordinates $(t, \boldsymbol{y}, \boldsymbol{v})$ and $(s, \boldsymbol{x}, \boldsymbol{u})$ are equivalent in the sense that they are related through a composition of transformations (1.5) and (1.6).

Now suppose the evolution equation (1.2), $\boldsymbol{\Phi} = \boldsymbol{0}$, is related to (2.1). It is obvious that $Y = \partial_s$ is a point symmetry operator for (1.2), hence for $\boldsymbol{\Omega} = \boldsymbol{0}$. From (1.3) and (1.10) we see that Y corresponds to the standard form symmetry $X(\boldsymbol{f})$ with

$$\boldsymbol{f} = \partial_s \boldsymbol{V} - \sum_j (\partial_s Y^j)\boldsymbol{v}_{y_j} - (\partial_s T)\boldsymbol{K} \tag{2.3}$$

and $\partial_s T \neq 0$.

Conversely, suppose Y, (2.2), is a point symmetry operator for $\boldsymbol{\Omega} = \boldsymbol{0}$, with $\tau \neq 0$. We will construct an evolution equation related to $\boldsymbol{\Omega} = \boldsymbol{0}$ for which Y is the time translation generator. Since Y is a symmetry operator it generates a flow on the coordinate space

$$\exp \alpha Y \colon (t, \boldsymbol{y}, \boldsymbol{v}) \to (t^*(\alpha), \boldsymbol{y}^*(\alpha), \boldsymbol{v}^*(\alpha))$$

which takes solutions of (2.1) to solutions, [1, 5, 6]. Indeed this flow is uniquely determined by the equations

$$\begin{aligned}
(\partial t^*/\partial \alpha)(\alpha, t, \boldsymbol{y}, \boldsymbol{v}) &= \tau(t^*, \boldsymbol{y}^*, \boldsymbol{v}^*) \\
(\partial \boldsymbol{y}^*/\partial \alpha)(\alpha, t, \boldsymbol{y}, \boldsymbol{v}) &= \boldsymbol{\xi}(t^*, \boldsymbol{y}^*, \boldsymbol{v}^*) \\
(\partial \boldsymbol{v}^*/\partial \alpha)(\alpha, t, \boldsymbol{y}, \boldsymbol{v}) &= \boldsymbol{\eta}(t^*, \boldsymbol{y}^*, \boldsymbol{v}^*)
\end{aligned}$$

with initial conditions

$$t^*(0)=t \qquad y^*(0)=y \qquad v^*(0)=v. \tag{2.4}$$

Let $v(t,y)$ be a solution of (2.1). It follows from (2.4) that for $|\alpha|$ small the equations

$$t^*(\alpha,t,y,v(t,y))=t^* \qquad y^*(\alpha,t,t,v(t,y))=y^* \tag{2.5}$$

can be solved for t and y. Substituting this result into the expression for v^* we obtain

$$v^*(\alpha,t^*,y^*)\equiv v^*(\alpha,t,y,v(t,y)). \tag{2.6}$$

Since Y is a symmetry operator it follows that $v^*(\alpha,t^*,y^*)$ is a solution of (2.1) [1, 5, 6]:

$$v^*_{t^*}=K(y^*,v^*,v^*_{i_1\ldots i_n}). \tag{2.7}$$

Furthermore a straightforward computation using (2.3) yields

$$\begin{aligned} v^*_\alpha(\alpha,t^*,y^*)&=\eta(t^*,y^*,v^*)-\sum_{j=1}^n \xi^j(t^*,y^*,v^*)v^*_{y^{j*}}\\ &\quad-\tau(t^*,y^*,v^*)K(y^*,v^*,v^*_{i_1\ldots i_n}). \end{aligned}\tag{2.8}$$

Thus the function $v^*(\alpha,t^*,y^*)$ satisfies two evolution equations simultaneously, i.e. the flows are commuting. Now set $t^*=c$, c constant, in (2.7). Since $\tau\neq 0$ we can solve the equation $t^*(\alpha,t,y,v)=c$ to obtain α as a function $\alpha=f(c,t,y,v)$. In place of our original coordinates (t,y,v) we have new coordinates (s,x,u) where

$$\begin{aligned} &s\equiv\alpha=f(c,t,y,v) \qquad x\equiv y^*=X(c,t,y,v)\\ &u\equiv v^*=V(c,t,y,v). \end{aligned}\tag{2.9}$$

Equation (2.7) is now meaningless but (2.8) takes the form

$$u_s=\eta(c,x,u)-\sum_{j=1}^n \xi^j(c,x,u)u_{x^j}-\tau(c,x,u)K(x,u,u_{i_1\ldots i_n}) \tag{2.10}$$

an evolution equation related to (2.1). Furthermore, $\partial_s=Y$.

An important feature of the preceding construction is that the related evolution equation (2.10) can be determined immediately from the original evolution equation (2.1) and the symmetry (2.2). There is no need to determine the coordinate transformation (2.9) and then change coordinates in (2.1). For example, the Hamilton-Jacobi equation

$$v_t=\sum_{i=1}^n v^2_{y_i}+W(y) \tag{2.11}$$

where the potential W is homogeneous of order -2, ($W(\lambda y)=\lambda^{-2}W(y)$), clearly admits the symmetry

$$Y=(2t-1)\partial_t+\sum_{i=1}^n y_i\partial_{y_i}. \tag{2.12}$$

It follows immediately from (2.10) that the related evolution equation is (for $c=0$)

$$u_s=\sum_{i=1}^n (u^2_{x_i}-x_iu_{x_i})+W(x). \tag{2.13}$$

The explicit coordinate transformation mapping (2.11) to (2.13) can be obtained by solving equations (2.3):

$$s=-\tfrac{1}{2}\ln(1-2t) \qquad x_i=\frac{y_i}{(1-2t)^{1/2}} \qquad u=v.$$

If $W(\boldsymbol{y})$ is invariant under the translations $y_i \to y_i+a$, $1 \leqslant i \leqslant n$, for all a then (2.11) admits the symmetry

$$Y'=-\partial_t+\sum_{i=1}^{n}\partial_{y_i}. \tag{2.14}$$

The related evolution equation is

$$u_s=\sum_{i=1}^{n}(u_{x_i}^2-u_{x_i})+W(\boldsymbol{x}). \tag{2.15}$$

If $W(\boldsymbol{y})$ is homogeneous of order +2 then (2.11) admits the symmetry

$$Y''=-\partial_t+\sum_{i=1}^{n}y_i\partial_{y_i}+2v\partial_v \tag{2.16}$$

with related evolution equation

$$u_s=\sum_{i=1}^{n}(u_{x_i}^2-x_iu_{x_i})+2u+W(\boldsymbol{x}). \tag{2.17}$$

The system of equations

$$\begin{aligned} v_t^1&=v_y^2\\ v_t^2&=\tfrac{1}{3}v_{yyy}^1+\tfrac{8}{3}v^1v_y^1 \end{aligned} \tag{2.18}$$

(equivalent to the Boussinesq equation) admits the scaling symmetry

$$Y=-2t\partial_t-y\partial_y+2v^1\partial_{v^1}+3v^2\partial_{v^2}. \tag{2.19}$$

The related evolution equation (for $c=\frac{1}{2}$) is

$$\begin{aligned} u_s^1&=2u^1+xu_x^1+u_x^2\\ u_s^2&=3u^2+xu_x^2+\tfrac{1}{3}u_{xxx}^1+\tfrac{8}{3}u^1u_x^1. \end{aligned} \tag{2.20}$$

The evolution equation

$$v_t=v_{yy}/v_y^2 \tag{2.21}$$

(equivalent to the heat equation by a 'trivial' coordinate transformation [6, p 316]) admits the symmetry

$$Y=-\partial_t-\tfrac{1}{2}yv\partial_y+t\partial_v \tag{2.22}$$

which corresponds to the related equation ($c=1$)

$$u_s=u_{xx}/u_x^2+1+\tfrac{1}{2}xuu_x. \tag{2.23}$$

Additional examples can be found in [2]. Note that a solution of (2.10) with $u_s=0$ yields a group-invariant solution of (2.1) [7, 8].

3. The equivalence classes

In general the association between related evolution equations and point symmetry operators is not strictly one to one because transformations of the form (1.5) or (1.6) may map one evolution equation to another and these transformations do not alter the symmetry operator. If the evolution equation (1.1) is allowed to be a rational or a general analytic function of the derivatives $\boldsymbol{v}_{i_1\ldots i_n}$, rather than a polynomial function, then an arbitrary transformation

$$t=s \qquad y=\boldsymbol{Y}(\boldsymbol{x},\boldsymbol{u}) \qquad \boldsymbol{v}=\boldsymbol{V}(\boldsymbol{x},\boldsymbol{u}) \tag{3.1}$$

will indeed map (1.1) to another evolution equation (1.2). However the polynomial restriction on the derivatives places severe limitations on these transformations. Indeed, for scalar evolution equations we shall show that necessarily $\partial Y_i/\partial u=0$, $1\leqslant i\leqslant n$, unless (1.1) is a first-order linear partial differential equation. (The *trivial* transformations

$$t=s \qquad y=\boldsymbol{Y}(\boldsymbol{x}) \qquad \boldsymbol{v}=\boldsymbol{V}(\boldsymbol{x},\boldsymbol{u})$$

always map evolution equations to evolution equations.) Even for analytic evolution equations which are not required to be polynomials in the derivatives $\boldsymbol{v}_{i_1\ldots i_n}$, the transformations

$$t=s+\varphi(\boldsymbol{x},\boldsymbol{u}) \qquad y=\boldsymbol{x} \qquad \boldsymbol{v}=\boldsymbol{u} \tag{3.2}$$

for non-constant φ will ordinarily not map these equations to other evolution equations. We shall derive a degeneracy condition on the evolution equation that is necessarily satisfied if the equation admits a non-trivial transformation (3.2). For scalar evolution equations we shall derive the precise conditions under which transformations (3.2) are admitted; in particular, $\partial\varphi/\partial u=0$ always unless (1.1) is a first-order linear or quadratic partial differential equation. Finally, we shall compose the transformations (3.1), (3.2) and show for scalar polynomial, rational and general analytic evolution equations the connection between related equations and Lie symmetries is one to one (modulo trivial coordinate transformations); the only exceptions being first-order linear and quadratic equations, and higher-order equations that are degenerate in the sense of theorems 4 and 5. For vector evolution equations the situation is much more complicated and we give only partial results.

We first determine the conditions under which the scalar evolution equation

$$\Omega\equiv v_t-K(y,v,v_{i_1\ldots i_n})=0 \tag{3.3}$$

a polynomial in the $M>0$ derivatives $v_{i_1\ldots i_n}$, is mapped to a similar evolution equation by a coordinate transformation of the form

$$t=s \qquad y=\boldsymbol{Y}(\boldsymbol{x},u) \qquad v=V(\boldsymbol{x},u) \tag{3.4}$$

$y=(y_1,\ldots,y_n)$, $\boldsymbol{x}=(x_1,\ldots,x_n)$, where

$$\det(\partial\boldsymbol{Y}(\boldsymbol{x},u(s,x))/\partial\boldsymbol{x})\neq 0.$$

Defining the $n\times n$ coordinate transformation matrix

$$A_{ij}\equiv(\partial u_i/\partial x_j)=(\partial_{x_j}Y_i+\partial_u Y_i u_{x_j}) \tag{3.5}$$

we see that

$$
\begin{aligned}
&v_{y_i} = A_{ij}^{-1}(\partial_{x_j} V + \partial_u V u_{x_j}) \\
&v_{y_i y_l} = B A_{ih}^{-1} A_{lk}^{-1} u_{x_h x_k} + (\text{terms involving first derivatives of } u) \\
&v_{y_i y_l y_m} = B A_{ih}^{-1} A_{lk}^{-1} A_{mj}^{-1} u_{x_h x_k x_j} \\
&\qquad + (\text{terms involving second and lower-order derivatives of } u) \\
&\ldots .
\end{aligned}
\tag{3.6}
$$

Here we sum over repeated indices on the right-hand sides of (3.6) and

$$B = (1-n)\partial_u V + A_{pj}^{-1}(\partial_u V \partial_{x_j} Y_p - \partial_{x_j} V \partial_u Y_p) = f(x, u)/\det A \tag{3.7}$$

$$\det A = \det(\partial_{x_j} Y_i) + \sum_{k=1}^{n} \det \begin{pmatrix} \partial_{x_1} \boldsymbol{Y} \\ \vdots \\ \partial_{x_{k-1}} \boldsymbol{Y} \\ \partial_u \boldsymbol{Y} \\ \partial_{x_{k+1}} \boldsymbol{Y} \\ \vdots \\ \partial_{x_n} \boldsymbol{Y} \end{pmatrix} u_{x_k}. \tag{3.8}$$

Also

$$v_t = \frac{u_t}{\det A} \det \left(\begin{array}{c|c} \partial_{x_j} Y_i & \partial_{x_j} V \\ \hline \partial_u Y_i & \partial_u V \end{array} \right). \tag{3.9}$$

Now suppose $\partial_u \boldsymbol{Y} \not\equiv 0$. Then $\det A$ must be functionally dependent on at least one u_{x_k}, for otherwise $\det A = 0$ which is impossible. Further each term A_{ij}^{-1} is a rational function of the derivatives u_{x_k}:

$$A_{ij}^{-1} = \left(\sum_{k=1}^{n} a_{ij}^{k}(x, u) u_{x_k} + b_{ij}(x, u) \right) (\det A)^{-1}.$$

Substituting expressions (3.4), (3.6), (3.7) and (3.9) into (3.3) we see that (3.3) will transform to another evolution equation if and only if $(\det A)K$ is a polynomial function of the derivatives of u.

Suppose the highest-derivative terms $v_{i_1 \ldots i_n}$ appearing in K are those of order $q \geqslant 2$ where $q = i_1 + \ldots + i_n$. Then if $(\det A)K$ is a polynomial in the derivatives of u it must take the form

$$(\det A)K = \sum_{k_1, \ldots, k_q = 1}^{n} c_{k_1, \ldots, k_q}(x, u) u_{x_{k_1} \ldots x_{k_q}} + (\text{lower order terms}). \tag{3.10}$$

Since the inverse of the transformation (3.4) maps the new evolution equation back to (3.3), it follows that K must be linear in the derivatives $v_{i_1 \ldots i_n}$ of order q:

$$K = \sum_{h_1, \ldots, h_q = 1} C_{h_1, \ldots, h_q}(y, v) v_{y_{h_1} \ldots y_{h_q}} + (\text{lower order terms}). \tag{3.11}$$

In order for (3.10) and (3.11) to hold there must exist a non-zero function $g(x, u)$ such that

$$\sum_{k_1, \ldots, k_q} c_{k_1, \ldots, k_q} A_{k_1 h_1} \ldots A_{k_q h_q} = C_{h_1, \ldots, h_q} g. \tag{3.12}$$

At least one term $c_{\bar{k}_1, \ldots, \bar{k}_q}$ is non-zero and the coefficient of $u_{x_{\bar{k}_1}} \ldots u_{x_{\bar{k}_q}}$ on the left-hand side of (3.12) is

$$c_{\bar{k}_1, \ldots, \bar{k}_q} \partial_u Y_{h_1} \ldots \partial_u Y_{h_q}.$$

Thus $\partial_u Y \equiv 0$, a contradiction.

It follows that if $\partial_u Y \not\equiv 0$ we must have $q=1$. Then it is evident from the first equation (3.6) that $(\det A)K$ is a polynomial in the u_{x_k} if and only if

$$(\det A)K = \sum_{k=1}^{n} c_k(x, u)u_{x_k} + d(x, u).$$

Thus,

$$K = \sum_{k=1}^{n} C_k(y, v)v_{y_k} + D(y, v) \tag{3.13}$$

and it is evident that every evolution equation of this form admits an arbitrary coordinate transformation (3.4).

Theorem 2. If a scalar polynomial evolution equation $v_t = K$ is mapped to another scalar polynomial evolution equation by a transformation

$$t = s \qquad y = Y(x, u) \qquad v = V(x, u)$$

with $\partial_u Y \not\equiv 0$ then K takes the form (3.13).

Next we consider the possibility that the transformation

$$t = s + \varphi(x, u) \qquad y = x \qquad v = u \tag{3.14}$$

will map the polynomial evolution equation

$$\begin{aligned} &\boldsymbol{\Omega} \equiv \boldsymbol{v}_t - \boldsymbol{K}(y, \boldsymbol{v}, \boldsymbol{v}_{i_1 \ldots i_n}) = 0 \\ &\boldsymbol{v} = (v^1, \ldots, v^m) \qquad y = (y^1, \ldots, y^n) \end{aligned} \tag{3.15}$$

to another polynomial evolution equation. Suppose the highest-derivative terms $v^j_{i_1 \ldots i_n}$ appearing in $\boldsymbol{K}$ are of order $q = i_1 + \ldots + i_n \geqslant 2$. A necessary condition that the transformation (3.14) map (3.15) to another evolution equation is

$$\partial \boldsymbol{\Omega} / \partial u_{j_1 \ldots j_n s^l} = 0$$

for all derivatives with $j_1 + \ldots + j_n + l = q$, $1 \leqslant l \leqslant q$. (This condition applies even if the requirement is dropped that $\boldsymbol{K}$ is a polynomial in the derivatives.) A straightforward computation yields the following lemma.

Lemma 1. A necessary condition that the evolution equation (3.15) is mapped to another evolution equation by the transformation (3.14) is

$$\sum_{\substack{q_i \geqslant k_i \\ \Sigma q_i = q}} \left[\frac{\partial K^{(h)}}{\partial v^k_{q_1 \ldots q_n}} - K^{(l)} \varphi_v \cdot \frac{\partial K^{(h)}}{\partial \boldsymbol{v}_{q_1 \ldots q_n}} \right] \binom{q_1}{k_1} \cdots \binom{q_n}{k_n} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}y_1} \right)^{q_1 - k_1} \cdots \left(\frac{\mathrm{d}\varphi}{\mathrm{d}y_n} \right)^{q_n - k_n} = 0 \tag{3.16}$$

for $1 \leqslant h$, $l \leqslant m$, and all $k_i \geqslant 0$ with $\Sigma_{i=1}^n k_i < q$. Here $\binom{q_i}{k_i}$ is a binomial coefficient.

In the scalar case ($m = 1$) this necessary condition reduces to

$$\sum_{\substack{q_i \geqslant k_i \\ \Sigma q_i = q}} \frac{\partial K}{\partial v_{q_1 \ldots q_n}} \binom{q_1}{k_1} \cdots \binom{q_n}{k_n} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}y_1} \right)^{m_1 - k_1} \cdots \left(\frac{\mathrm{d}\varphi}{\mathrm{d}y_n} \right)^{m_n - k_n} = 0 \tag{3.17}$$

for all $k_i \geqslant 0$ with $\Sigma_{i=1}^n k_i < q$.

We now treat the scalar case in detail to derive necessary and sufficient conditions that

$$t = s + \varphi(x, u) \qquad y = x \qquad v = u \tag{3.18}$$

map a scalar evolution equation to an evolution equation. Suppose first that $\partial\varphi/\partial u \neq 0$. Making use of a trivial transformation (1.4) we can assume that (3.18) takes the simple form

$$t = s + u \qquad y = x \qquad v = u. \tag{3.19}$$

Thus,

$$v_t = \frac{u_s}{1+u_s} \qquad v_{y_i} = \frac{u_{x_i}}{1+u_s} \qquad \partial_{y_i} = \partial_{x_i} - \frac{u_{x_i}}{1+u_s}\partial_s. \tag{3.20}$$

Now suppose the highest derivative terms in K are of order $q \geqslant 2$. Then with the substitution (3.19) a necessary condition for the evolution equation $\Omega = 0$ to be mapped to another evolution equation is that the coefficients of each of the derivatives $u_{j_1 \ldots j_n s^l}$ vanish where either $l > 1$ or $l = 1$, $j_1 + \ldots + j_n > 0$. Suppose this necessary condition is satisfied. Then in the expression of all derivatives $v_{i_1 \ldots i_n}$ in K as u derivatives we can set equal to zero all terms involving the $u_{j_1 \ldots j_n s^l}$. Thus we obtain

$$v_{i_1 \ldots i_n} \sim \frac{u_{i_1 \ldots i_n}}{1+u_s} \tag{3.21}$$

and the evolution equation $\Omega = 0$ takes the form

$$\frac{u_s}{1+u_s} - \sum_{k=0}^{N} \frac{M_k}{(1+u_s)^k} = 0$$

where M_k is a homogeneous polynomial of order k in the derivatives $u_{i_1 \ldots i_n}$ (with coefficients that are functions of x, u) and $M_N \neq 0$. Thus,

$$u_s(1+u_s)^{N-1} - \sum_{k=0}^{N} M_k(1+u_s)^{N-k} = 0 \tag{3.22}$$

and this is equivalent to an evolution equation $u_s - J = 0$ if and only if the coefficients of u_s^l, $l > 1$, vanish identically in (3.22) and the coefficient of u_s is a non-zero function $f(x, u)$. (Indeed if (3.22) has multiple roots as a polynomial in u_s then any one root $u_s = J_h$ yields a polynomial in the derivatives $u_{i_1 \ldots i_n}$ which is of strictly lower order than that in $v_t = K$. However, the inverse transformation from u to v cannot possibly increase the order of J_h. Thus (3.22) must have a single root of multiplicity one.) Since the terms M_k are either strictly independent or identically zero, it follows that there are only two possibilities:

$$\begin{aligned} &\text{(i)} \qquad u_s - \frac{M_1 + M_0}{M_0 - 1} = 0 \\ &\text{(ii)} \qquad u_s + M_2 + 1 = 0. \end{aligned} \tag{3.23}$$

It is evident that case (i) cannot occur for $q > 1$ because condition (3.17) with $\mathrm{d}\varphi/\mathrm{d}y_i = v_{y_i}$, $K = M_1/(1+u_s) + M_0$ requires that K depend linearly on the v_{y_i}, which is impossible. Case (ii) with $K = M_2/(1+u_s)^2 + 1$ is more difficult to rule out. Conditions (3.17) imply that

$$K = \sum_{k, i_1 + \ldots + i_n = q} a_{i_1, \ldots, i_n, k}(y, v) v_{i_1 \ldots i_n} v_{y_k} + b(y, v).$$

Substitution of this expression into (3.17) and some tedious algebra shows that $a_{i_1, \ldots, i_n, k} \equiv 0$.

If $q=1$ then condition (3.17) no longer applies and cases (3.23(i), (ii)) occur in complete generality where the M_i are homogeneous polynomials of degree i in the first derivatives

$$\begin{aligned}\Omega &\equiv v_t - M_2(y, v, v_{y_i}) - 1 = 0\\ \Omega &\equiv v_t - M_1(y, v, v_{y_i}) - M_0(y, v) = 0 \qquad M_0 \neq 1.\end{aligned} \tag{3.24}$$

Next we derive necessary and sufficient conditions that the transformation (3.18) map a scalar polynomial evolution equation to a polynomial evolution equation where $\partial\varphi/\partial u = 0$, $\partial\varphi/\partial \boldsymbol{x} \neq 0$. Making use of a trivial transformation (1.4) we can assume that (3.18) takes the form

$$t = s - x_1 \qquad y = x \qquad v = u. \tag{3.25}$$

It follows that

$$\begin{aligned}&v_t = u_s \qquad v_{y_i} = u_{x_i} + \delta_{1i} u_s \qquad \partial_{y_i} = \partial_{x_i} + \delta_{1i}\partial_s \qquad v_{y_{k_1}\cdots y_{k_p}} = u_{x_{k_1}\cdots x_{k_p}}\\ &v_{y_{k_1}\cdots y_{k_p} y_1^l} = \sum_{h=0}^{l} \binom{l}{h} u_{x_{k_1}\cdots x_{k_p} x_1^h s^{l-h}} \qquad k_j \neq 1 \qquad l \geq 1.\end{aligned} \tag{3.26}$$

Now suppose the highest derivative terms in K are of order $q' \geq 2$. Then from (3.17) with $\mathrm{d}\varphi/\mathrm{d}y_i = -\delta_{i1}$ we find $\partial K/\partial v_{q_1\ldots q_n} = 0$ if $\Sigma q_i = q'$ and $q_1 > 0$. Since all qth order derivatives of v can be expressed in terms of qth order derivatives of u, if $q'-1 \geq 2$ we can apply (3.17) again for $q = q'-1$ and conclude that $\partial K/\partial v_{q_1\ldots q_n} = 0$ if $\Sigma q_i = q'-1$ and $q_1 > 0$. Continuing in this fashion we find that a necessary condition for the transformation (3.25) to map $\Omega = 0$ into another evolution equation is $\partial K/\partial v_{q_1\ldots q_n} = 0$ whenever $q_1 > 0$ and $\Sigma q_i \geq 2$. It follows easily that the necessary and sufficient condition for (3.25) to map $\Omega = 0$ into a polynomial evolution equation is that this equation take the form

$$\Omega \equiv v_t - K'(y, v, v_{y_{k_1}\cdots y_{k_p}}) - a(y, v) v_{y_1} = 0 \tag{3.27}$$

where $k_i \neq 1$ and $a \neq 1$. If $q = 1$ this same necessary and sufficient condition holds with $p = 1$.

Theorem 3. If a scalar polynomial evolution equation $\Omega = v_t - K = 0$ is mapped to another scalar polynomial evolution equation by a transformation

$$t = s + \varphi(y, u) \qquad y = x \qquad v = u$$

with $\mathrm{d}\varphi \neq 0$ then Ω is equivalent via a 'trivial' transformation to one of the canonical forms (3.24) or (3.27).

The most general possible coordinate transformation mapping a scalar polynomial evolution equation to another such equation (without changing the time symmetry operator) can be expressed as the composition of the transformations (3.4) and (3.18). A tedious but straightforward argument using the techniques already introduced in this section shows that this composition leads to no 'non-trivial' mappings between evolution equations other than those already discovered.

Theorem 4. The only scalar polynomial evolution equations which can be mapped to other scalar polynomial evolution equations by 'non-trivial' transformations

$$t = s + \varphi(\boldsymbol{x}, u) \qquad \boldsymbol{y} = \boldsymbol{Y}(\boldsymbol{x}, u) \qquad v = V(\boldsymbol{x}, u)$$

are equivalent via 'trivial' transformations

$$t = s' \qquad y = Y'(x') \qquad v = V(x', u')$$

to one of the canonical forms (3.24) or (3.27).

We next examine the effect on these results of permitting K in the scalar evolution equations $\Omega \equiv v_t - K = 0$ to be a *rational* function of the spatial derivatives $v_{i_1 \ldots i_n}$. Now the transformations

$$t = s \qquad y = Y(x, u) \qquad v = V(x, u) \tag{3.28}$$

are 'trivial' since all these transformations map a scalar rational evolution equation to a scalar rational evolution equation. The possible 'non-trivial' transformations are those of the form

$$t = s + \varphi(x, u) \qquad y = x \qquad v = u \tag{3.29}$$

with $d\varphi \not\equiv 0$. Using a suitable transformation (3.28) we can assume that (3.29) takes the form (3.25):

$$t = s - x_1 \qquad y = x \qquad v = u.$$

Applying an argument used earlier in this section we can show that $\partial K / \partial v_{q_1 \ldots q_n} = 0$ for $q_1 > 0$ and $\Sigma\, q_i \geqslant 2$ if $\Omega = 0$ admits the transformation (3.2). The transformed equation thus takes the form

$$u_s = \left(\sum_{h=0}^{p} P_h(x, u, u_{j_2 \ldots j_n})(u_1 + u_s)^h \right) \left(\sum_{l=0}^{q} Q_l(x, u, u_{j_2 \ldots j_n})(u_1 + u_s)^l \right)^{-1} \tag{3.30}$$

where P_h, Q_l are polynomial functions of the derivatives $u_{j_2 \ldots j_n}$, $P_p Q_q \neq 0$ and $|P_0| + |Q_0| \neq 0$. Multiplying both sides of (3.30) by the denominator of the right-hand side we see that the resulting expression defines a rational evolution equation if and only if the coefficients of u_1^k for $k > 1$ are identical on each side. The result is the theorem below.

Theorem 5. The only scalar rational evolution equations which can be mapped to other rational evolution equations by 'non-trivial' transformations (3.29) are those equivalent via 'trivial' transformations (3.28) to one of the canonical forms

(i) $\qquad v_t = P_0 + P_1 v_{y_1} \qquad p = 1 \qquad q = 0$

(ii) $\qquad v_t = \dfrac{P_0 + P_1 v_{y_1} + P_2 v_{y_1}^2}{Q_0 + P_2 v_{y_1}} \qquad p = 2 \qquad q = 1.$

Finally we examine the effect of permitting K in the scalar evolution equations $\Omega \equiv v_t - K = 0$ to be a *local analytic* function of the spatial derivatives $v_{i_1 \ldots i_n}$. Again the transformations (3.28) are 'trivial' and the transformations (3.29) are 'non-trivial'. By applying a suitable transformation (3.28) we can assume that (3.29) takes the simple form (3.25).

Theorem 6. The only scalar analytic evolution equations which can be mapped (locally) to other analytic evolution equations by 'non-trivial' transformations (3.29) are those equivalent via 'trivial' transformations (3.28) to an equation in the canonical form

$$v_t - K(y, v, v_{i_2 \ldots i_n}, v_{y_1}) = 0.$$

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